

Six alternative proofs of the three-squares problem

Alon Pinto, Weizmann Institute of Science

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The three-squares problem, often attributed to Gardner (1970) is the following:

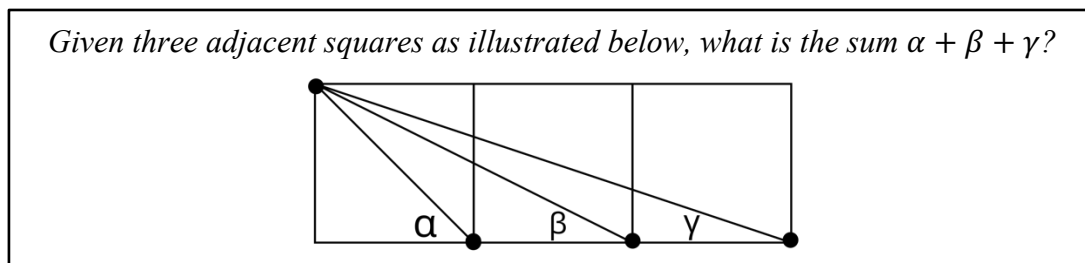


Figure 1: The three-squares problem

Other versions of the problem may ask directly for a proof that $\alpha + \beta + \gamma = \frac{\pi}{2}$. The three-squares problem has many elementary solutions that draw on various mathematical fields, a property that makes it popular in the context of teaching multiple proof tasks (Hoseana, 2024). This document presents 6 solutions that are adaptations of proofs presented by students in an Israel advanced-track 11-grade class. A video recording of the lesson (in Hebrew) can be found [VIDEO-LM archive](#) (Karsenty & Arcavi, 2017).

Proof 1 – planar geometry

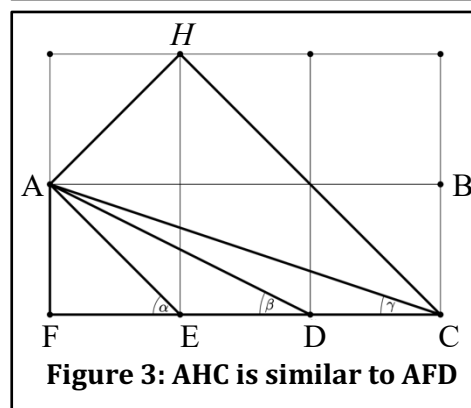
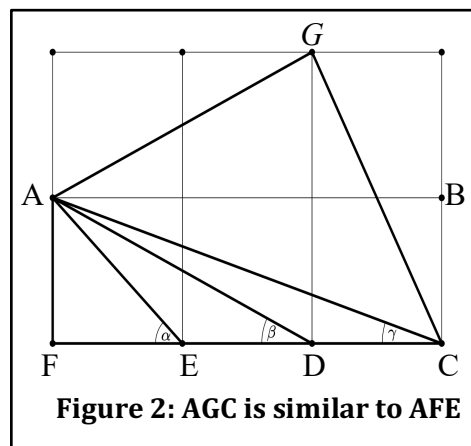
Add three auxiliary squares and connect the points A and C to the point G, as in Figure 2. The triangles ACG and AFE are right angled isosceles and hence similar and thus $\angle ACG = \alpha$. Since $\angle GCB = \beta$, we can conclude that $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Proof 2 – planar geometry (similar triangles)

The triangles AED and CEA are similar since the two triangles share the angle AEC and since $\frac{AE}{ED} = \sqrt{2} = \frac{CE}{EA}$. It follows that $\angle EAC = \angle EDA = \beta$, and since γ and $\angle CAB$ are alternating angles, $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Proof 3 – planar geometry

Similarly to Proof 1, add three auxiliary squares and two edges to a point H as illustrated in Figure 3. The new right-angled triangle AHC is similar to



AFD with a similarity factor of $\sqrt{2}$. It follows that $\angle ACG = \beta$ and since $\angle BCG = \alpha$ we can conclude that $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Proof 4 - Trigonometry

First, note that $\alpha = \arctan\left(\frac{1}{1}\right)$, $\beta = \arctan\left(\frac{1}{2}\right)$, and $\gamma = \arctan\left(\frac{1}{3}\right)$.

The sum formula for tangent:

$$\tan(\beta + \gamma) = \frac{\tan(\beta) + \tan(\gamma)}{1 - \tan(\beta)\tan(\gamma)}$$

This implies that $\tan(\beta + \gamma) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1$ or that $\arctan(\beta + \gamma) = 1$ and because all angles are acute, we can conclude that $\beta + \gamma = \frac{\pi}{4}$.

Proof 5 – Analytic geometry

We represent the sum $\beta + \gamma$ as the angle between the line equations $y = \frac{x}{2}$ and $y = -\frac{x}{3}$. Note that both β and γ are less than $\frac{\pi}{4}$ and thus $\beta + \gamma < \frac{\pi}{2}$. We can therefore apply the formula for the angle between lines and conclude that $\tan(\beta + \gamma) = \left| \frac{\frac{1}{2} - (-\frac{1}{3})}{1 + (-\frac{1}{6})} \right| = 1$ and similarly to Proof 4 it follows that $\beta + \gamma = \frac{\pi}{4}$.

Proof 6 – Complex numbers

Let $Z = 2a + ia = \sqrt{5}e^{i\alpha}$ and $W = 3a - ia = \sqrt{10}e^{-i\beta}$ then

$$\beta + \gamma = \arg(ZW^{-1}) = \arg\left(\frac{Z\bar{W}}{|W|}\right)$$

$$\frac{Z\bar{W}}{|W|} = \frac{1}{\sqrt{10}}(2a + ia)(3a + ia) = \frac{1}{\sqrt{10}}(5a^2 + i5a^2)$$

It follows that $\beta + \gamma = \arg(5a^2 + i5a^2) = \arctan\left(\frac{5}{5}\right) = \frac{\pi}{4}$ (see Proof 4).

References

Hoseana, J. (2024). Using Gardner's Three-Squares Problem for a Group Project in a Mathematical Problem Solving Module. *The Mathematics Enthusiast*, 21(1), 467-484.

Gardner, M. (1970). Mathematical games: nine new puzzles to solve, some answers and addenda. *Scientific American*, 222(6), 112–115.

Karsenty, R., & Arcavi, A. (2017). Mathematics, lenses and videotapes: A framework and a language for developing reflective practices of teaching. *Journal of mathematics teacher education*, 20, 433-455.